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Large deflections of nonlinearly elastic non-prismatic cantilever beams made from materials obeying the generalized Ludwick constitutive law

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Abstract This paper studies large deflections of nonlinearly elastic cantilever beams made from materials obeying the generalized Ludwick constitutive law. An exact moment-curvature formula which can be applied to study arbitrarily loaded and supported beams of rectangular cross-sections is developed. Several advantages of the generalized Ludwick's model are illustrated. Numerical examples considered in this materially and geometrically nonlinear analysis clearly indicate rich nonlinear behavior of the beams.

Keywords Large deflections · Generalized Ludwick constitutive law · Non-prismatic beams · Material and geometrical nonlinearities

1 Introduction

In general, when light-weight structures are made of slender structural elements, these elements can easily be deformed into states with large deflections within the range of small strains. Hence, a geometrically nonlinear analysis has to be performed to derive the equations of equilibrium. Furthermore, when such slender elements are made from e.g. rubber or rubber-like materials which have nonlinear stress–strain relations, also material nonlinearities have to be considered. Geometrically and materially nonlinear analysis thus often accompanies development of many engineering applications such as car tyres, bridge and engine mountings, structural dampers, ring seals, tennis and basket balls, etc.

There have been a large number of contributions pertaining to nonlinear analysis of structural elements, of which the majority consider only the geometrical nonlinearities, e.g. [1–4]. However, contributions that consider both—the geometrical and material nonlinearities are not that frequent. A comprehensive list of publications on this topic is given in the References section, [5–21].

Contributions that are most relevant to the problem addressed here are briefly discussed below. Lee [11] and Eren [20] studied the large deflections of a prismatic cantilever beam made of Ludwick type material under a combined loading consisting of a uniformly distributed load and vertical concentrated force at the free end. Baykara et al. [13] investigated the effect of bimodulus material behavior on the horizontal and vertical deflections at the free end of a thin cantilever beam under an end moment. Brojan et al. [16] analyzed bending of non-prismatic shaped beams made of Ludwick type material with different stress–strain relationship in tensile and compressive domain. An analytical solution in terms of infinite series was found in the case when non-linear stress-strain relationship in

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tensile and compressive domain was identical. Similar problem was also examined by Shatnawi et al. [18]. Solano-Carrillo [21] considered the problem stated by Lee [11], with emphasis on the bending moment formulation.

By definition, an elastic material is one that exhibits complete and immediate recovery to the undeformed configuration from an imposed displacement state on release of the load. This physical phenomenon is exhibited by many materials when strains are sufficiently small. It is often observed that for many elastic materials a reasonable approximation regarding the stress-strain relationship at small strains is Hooke's law,

$$\sigma(\varepsilon) = \mathrm{sign}(\varepsilon) E|\varepsilon|, \tag{1}$$

i.e. relation between stress and strain is linear. The elements σ, ε, and E in (1) are stress, strain, and material constant (modulus of elasticity), respectively, and sign is the sign function, so that $\mathrm{sign}(\varepsilon) = -1$ for $\varepsilon < 0$, i.e. in the case of compression, and $\mathrm{sign}(\varepsilon) = 1$ for $\varepsilon \geq 0$, i.e. in the case of tension.

As the stresses are increasing, nonlinear response of the material is usually observed. To find a suitable law expressing the nonlinear stress-strain relations which would be sufficiently simple to allow considerable mathematical development, and would at the same time express the known behavior of as wide a range of materials as possible, is an important issue in nonlinear elasticity. It is necessary then to strike a compromise between mathematical tractability and applicability, [22]. One of the generalizations of the Hooke's law which describes nonlinearly elastic behavior of a number of highly elastic materials in a relatively clear and simple way is Ludwick's law,

$$\sigma(\varepsilon) = \mathrm{sign}(\varepsilon) E|\varepsilon|^{1/k}, \tag{2}$$

where E and k, $k > 0$, represent material constants by which the nonlinear behavior of the material is characterized. One can see that Ludwick's law corresponds to the Hooke's law by setting $k = 1$. It should be emphasized that expression (2) has one major shortcoming. Namely, the stress gradient goes to infinity for $k > 1$, and goes to zero for $k < 1$ when the strain value reaches zero, cf. Fig. 1. To overcome this problem, since no such material exists in nature, Jung and Kang [12] suggested a modified (generalized) form of the Ludwick's law, mathematically described by the following expression,

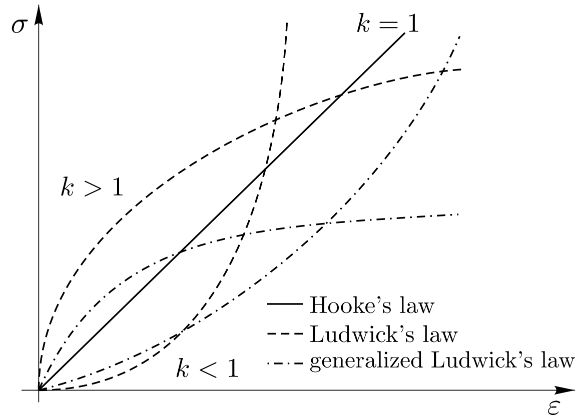

Fig. 1 Stress–strain relations in tensile domain

$$\sigma(\varepsilon) = \mathrm{sign}(\varepsilon) E\left[(|\varepsilon| + \varepsilon_0)^{1/k} - \varepsilon_0^{1/k}\right], \tag{3}$$

in which an additional parameter ε_0 is supplemented.

It is obvious that setting $\varepsilon_0 = 0$ leads to Ludwick's law and further, by setting $k = 1$ Hooke's law is obtained. Evidently, the generalized Ludwick's law is a three-parametric law, i.e. three parameters (E, k, and ε_0) are used to approximate the σ-ε diagrams obtained by experiments, whereas Ludwick's and Hooke's rheological models are two, and one-parametric laws, respectively.

The focus of this paper is on large deflection analysis of nonlinearly elastic, non-prismatic cantilever beams made from materials which obey the three-parametric generalized Ludwick rheological model.

2 Formulation of the problem

The basic assumptions made in the formulation of the problem studied are as follows:

- the beam is made of incompressible, homogeneous, isotropic, nonlinearly elastic material obeying the generalized Ludwick constitutive law;
- Bernoulli hypothesis, which states that plane cross-sections, which are perpendicular to the neutral axis before deformation, remain plain and perpendicular to the neutral axis after deformation and do not change their shape and area, is valid;
- shear stresses are negligible in comparison with the normal stresses because the length-to-height ratio

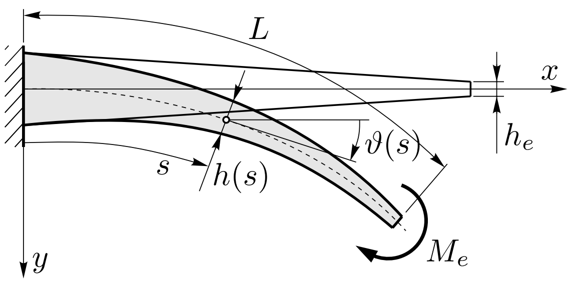

Fig. 2 Pure bending of non-prismatic cantilever beam

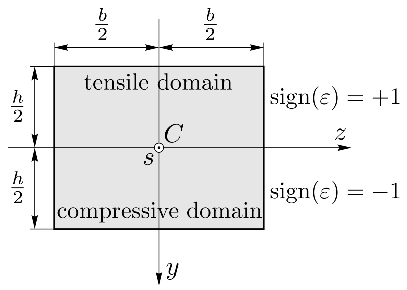

Fig. 3 Cross-section of the cantilever beam

of the column is large, i.e. the change of length in any linear element in the material is small compared with the length of the element in the undeformed state.

Let us now consider a slender, initially straight cantilever beam of length L subjected to moment M_e which is applied at the free end of the beam as illustrated in Fig. 2. The Cartesian coordinate system is chosen in such a manner that the abscissa axis coincides with the centroidal axis of the undeformed beam and the coordinate origin is fixed at the clamped end of the beam.

Let s, $0 \leq s \leq L$, denote the curvilinear coordinate along the axial line measured from the clamped end and $\vartheta(s)$ the angle between the positive part of the x-axis and the tangent to the neutral axis at point s. The cross-section of the beam is assumed to be of rectangular shape with constant width b and variable height $h(s)$, see Fig. 3.

A longitudinal shape of the beam is defined by the following equation

$$h(s) = h_e\left(\frac{1-p}{L}s + p\right)^q, \tag{4}$$

where p, q are given shape coefficients and h_e is the height of the beam at the free end. A family of longitudinal shapes of the beam defined by (4) is depicted in Fig. 4.

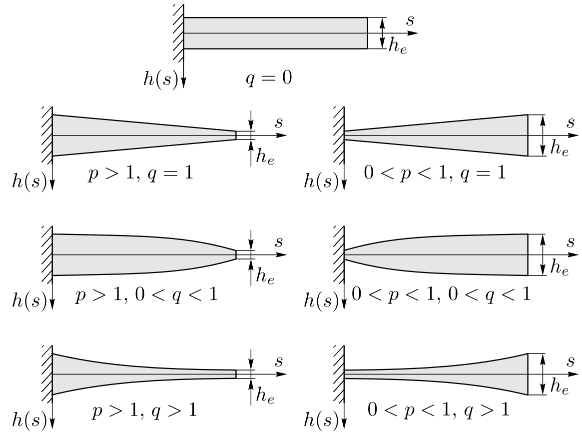

Fig. 4 Longitudinal shapes of the beam defined by (4)

3 Governing equations of the problem

The concepts and assumptions stated in previous sections serve as a starting point for the derivation of governing equations. It is known that the inner bending moment, acting at any cross-section of the beam, can be expressed with normal stress σ, as $M = -\int_A \sigma y dA$, where σ is related to the corresponding strain, see (3). Furthermore, taking into account the normal strain-curvature expression $\varepsilon = -y\rho^{-1}$ and after some work, the following expression for the inner bending moment-radius of curvature for the non-prismatic, nonlinearly elastic cantilever beams of rectangular cross-section made from materials obeying the generalized Ludwick constitutive law is deduced:

$$M(s) = 2bkE\rho(s)^2\left[\frac{(1+k)h(s) - 2k\varepsilon_0\rho(s)}{2(1+k)(1+2k)\rho(s)} \times \left(\frac{h(s)}{2\rho(s)} + \varepsilon_0\right)^{1+\frac{1}{k}} + \frac{k\varepsilon_0^{2+\frac{1}{k}}}{(1+k)(1+2k)} - \frac{h(s)^2\varepsilon_0^{\frac{1}{k}}}{8k\rho(s)^2}\right], \tag{5}$$

for all s, $0 \leq s \leq L$, where $h(s)$ is defined by (4). This formula is valid for arbitrary loading conditions, i.e. for cases where $M(s) \neq const$. Since radius of curvature $\rho(s)$ in (5) is given implicitly, one of the numerical methods should be used to calculate its value at each s, $0 \le s \le L$.

Note 1. Setting $\varepsilon_0 = 0$ (and further $k = 1$), yields an already known expression for inner bending moment-radius of curvature for Ludwick's (Hooke's) material. Indeed, for $\varepsilon_0 = 0$

$$M(s) = 2^{-\frac{1+k}{k}} E \frac{k}{1+2k} bh(s)^{\frac{1+2k}{k}} \rho(s)^{-\frac{1}{k}}, \tag{6}$$

or $M(s) = EI_k \rho(s)^{-\frac{1}{k}}$, where $I_k = 2^{-\frac{1+k}{k}} \frac{k}{1+2k} bh(s)^{\frac{1+2k}{k}}$ has been introduced, see also e.g. (2) from [11] and (11) from [17]. The radius of curvature can be determined from preceding equations explicitly, i.e. $\rho(s) = (\frac{EI_k}{M(s)})^k$. Similarly, by further setting $k = 1$, i.e. in a case of linearly elastic material, then $M(s) = EI\rho(s)^{-1}$, where $I = \frac{bh(s)^3}{12}$, and thus $\rho(s) = \frac{EI}{M(s)}$.

Note 2. As already mentioned, a reasonable approximation regarding the stress–strain relationship at sufficiently small strains for many elastic materials is Hooke's law. This also holds for generalized Ludwick type materials. Namely, the initial slope of the σ–ε curve for the generalized Ludwick material is

$$\left.\frac{d\sigma}{d\varepsilon}\right|_{\varepsilon=0} = \frac{\varepsilon_0^{\frac{1-k}{k}} E}{k} =: E_m, \tag{7}$$

where E_m corresponds to the elastic modulus of a linearly elastic material which has approximately the same response at small strains. Hence

$$M(s) = E_m I \rho(s)^{-1}. \tag{8}$$

This approximation may often be useful when e.g. searching for critical buckling force. The critical buckling force for a clamped-free column made from generalized Ludwick type materials is thereby simply $P_{cr} = \frac{\pi^2 \varepsilon_0^{(1-k)/k} EI}{4kL^2}$.

To calculate large deflections, exact (nonlinear) geometrical expressions should be considered. The Cartesian coordinates for a point on the neutral axis of the deflected non-linear cantilever beam (for all s, $0 \le s \le L$) can be calculated by employing exact expression for curvature $\rho(s)^{-1} = \vartheta'(s)$, where $'$ denotes the differentiation with respect to s, boundary conditions

$$\begin{aligned} &\vartheta(s=0) = 0, \\ &y(s=0) = 0, \\ &x(s=0) = 0, \end{aligned} \tag{9}$$

and following geometrical relations

$$\begin{aligned} &y'(s) = \sin\vartheta(s), \\ &x'(s) = \cos\vartheta(s). \end{aligned} \tag{10}$$

4 Numerical examples

To point out the differences between linear and both nonlinear constitutive relationships clearly and simply we have chosen the case of pure bending, i.e. the inner bending moment is constant along the longitudinal axis of the beam, namely $M(s) = M_e = const.$ for all s, $0 \le s \le L$. Nevertheless, if more general loading conditions are to be considered (thereby inner bending moment $M(s) \ne const.$), expression (5) has to be differentiated with respect to s and a corresponding expression for inner shear force has to be applied. In this case considerably more numerical effort is needed.

In the following numerical examples, cantilever beams of different longitudinal shapes are subjected to several different bending moments applied at the free end of each beam. The elastic constant for the generalized Ludwick's law E is defined in such a manner that $E = kE_m\varepsilon_0^{(k-1)/k}$, cf. (7), where E_m is the modulus of elasticity of a linearly elastic material which has approximately the same response at small strains. In addition, practical examples of Ludwick type material are annealed copper and N.P.8 aluminum alloy, characterized by material constants $E = 458.501$ MPa, $k = 2.160$, and $E = 455.743$ MPa, $k = 4.785$, respectively, [8, 9].

4.1 Example 1

The case of the prismatic cantilever beam ($h(s) = h_e = const.$) was analyzed first. The parameters that determine the shape are: $L = 1.0$ m, $b = 0.05$ m, $h_e = 0.025$ m, $q = 0$. Material parameters are as follows: $E_m = 70.0$ MPa, $k = 1.5$, $\varepsilon_0 = 0.07$, and $E = 43.273$ MPa. In this case the

Table 1 Comparison of radii of curvature ρ (mm)

M_e (kN m)	1	10	100	200	400	600
Generalized Ludwick's	4535.17	435.212	33.0120	13.7097	5.40785	3.07637
Ludwick's	103833.	3283.48	103.833	36.7104	12.9791	7.06492
Hooke's	4557.29	455.729	45.5729	22.7865	11.3932	7.59549

Table 2 Comparison of maximum strains $\varepsilon_{\max}$ (%)

M_e (kN m)	1	10	100	200	400	600
Generalized Ludwick's	0.27562	2.87216	37.8650	91.1764	231.145	406.324
Ludwick's	0.01204	0.38069	12.0386	34.0503	96.3088	176.931
Hooke's	0.27429	2.74286	27.4286	54.8571	109.714	164.571

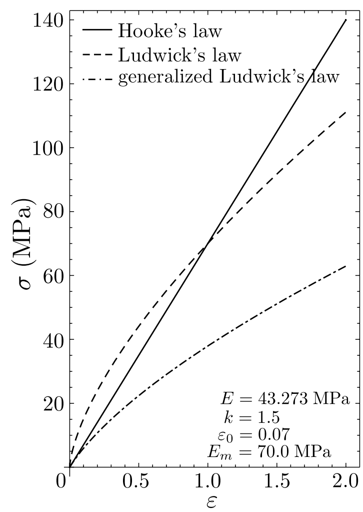

Fig. 5 Stress–strain curves for Hooke, Ludwick and generalized Ludwick type elastic material

radius of curvature ρ and hence maximum strain $\varepsilon_{\max} = \varepsilon(y = \pm h_e/2)$ were both constant for all s, $0 \le s \le L$. The values of ρ and $\varepsilon_{\max}$ at different end moments M_e are given in Tables 1 and 2, respectively.

The results listed in Table 1 and Table 2 can be interpreted by comparison of the stress–strain diagrams in Fig. 5. It is evident that beam made from Ludwick's material at small stress is less strained and therefore has larger radius of curvature than that made from Hooke's or generalized Ludwick's material. One can see from Fig. 5 and Tables 1 and 2 that the generalized Ludwick's material can be modeled with linearly elastic material at smaller loads, in our case e.g. up to $M_e \approx 10$ kN m or at smaller strains e.g. up to $\varepsilon_{\max} \approx 2.7\%$.

Moreover, it can also be observed that up to some characteristic strain Ludwick's material behaves as harder than Hooke's material. This characteristic strain can be determined by equating the elastic strain energies of both materials, i.e. $U_{\text{Hooke}} = U_{\text{Ludwick}}$. Namely, for elastic materials (Green or hyperelastic) there exists a strain energy function u, defined per unit volume, such that $u = \int_\Gamma \sigma(\varepsilon) d\varepsilon$ for arbitrary deformation along a path Γ. Therefore the total strain energy is $U = \int_{\mathcal{V}} u dV$, where $\mathcal{V}$ is the region occupied by the body. Bearing that in mind one can write

$$\int_{\mathcal{V}} \left(\int_\Gamma \operatorname{sign}(\varepsilon) E |\varepsilon| d\varepsilon \right) dV = \int_{\mathcal{V}} \left(\int_\Gamma \operatorname{sign}(\varepsilon) E |\varepsilon|^{1/k} d\varepsilon \right) dV. \tag{11}$$

Employing normal strain-curvature expression $\varepsilon = -y\rho^{-1}$ and after integration it can be written from (11) that

$$\rho = 2^{\frac{1-2k}{k-1}} 3^{\frac{k}{1-k}} k^{\frac{2k}{1-k}} (1+k)^{\frac{k}{k-1}} (1+2k)^{\frac{k}{k-1}} h.$$

Thus, the characteristic strain is found when both materials are subjected to $M_e = 897.012$ kN m and have $\rho = 5.080$ mm or equivalently $\varepsilon_{\max} = 246.038\%$. After this strain the Ludwick type material behaves as the softer one. This kind of behavior is typically met in nonlinear analysis, i.e. when material and geometrical nonlinearities are considered.

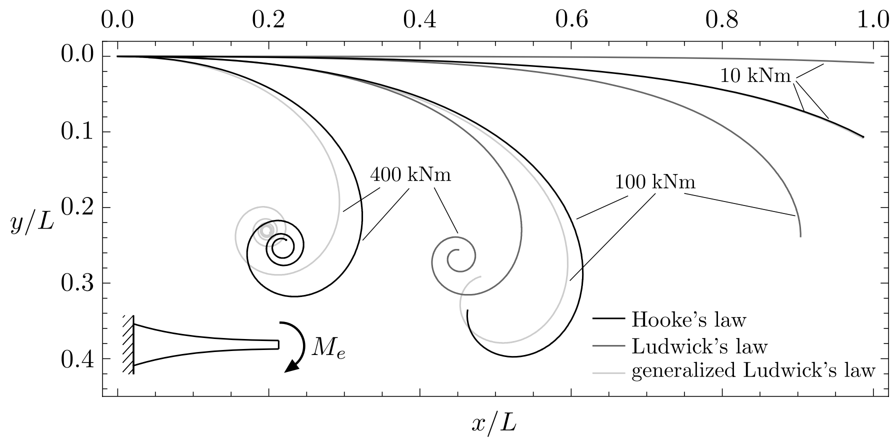

Fig. 6 Large deflections for example 2

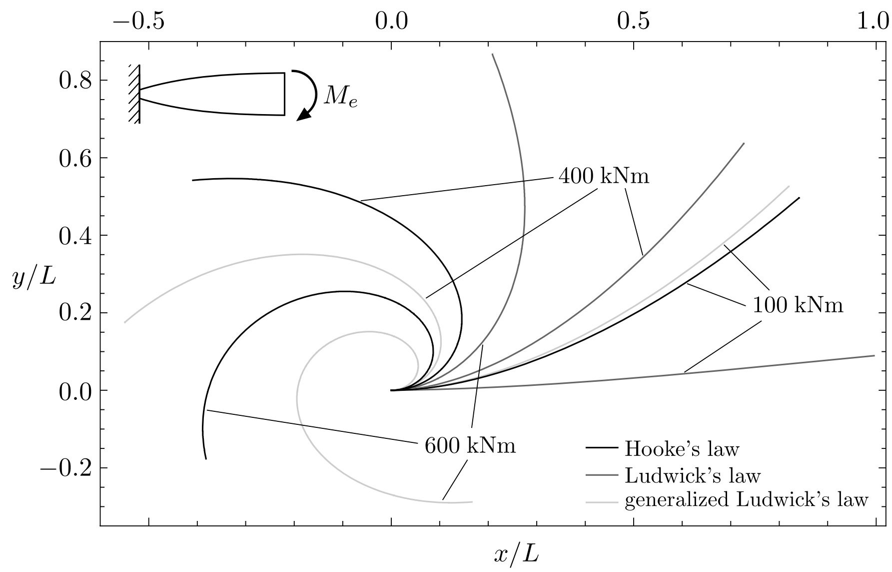

Fig. 7 Large deflections for example 3

4.2 Example 2

As the second example, a case of non-prismatic cantilever beam is considered. All the parameters are kept the same as in the first example except p and q which are now 1.4 and 4.0, respectively. The radius of curvature is no longer constant because the shape of the cantilever is non-prismatic, cf. (4), and now varies with s, $0 \le s \le L$. Displacement diagrams for this case are presented in Fig. 6.

At smaller loads the generalized Ludwick's material can be approximated with Hooke's material, as in the first example, since deflection curves in Fig. 6 are practically identical for $M_e = 10$ kN m.

4.3 Example 3

This example is similar to the second one, except that in this case $h_e = 0.2$ m, $p = 0.2$ and $q = 0.4$. Displacement diagrams are depicted in Fig. 7.

The same analysis applied on previous examples can be used to study large deflections of cantilever beams which have arbitrary longitudinal shapes and are made of nonlinearly elastic material which obeys the generalized Ludwick's constitutive law.

5 Conclusions

It is usually difficult to find a suitable mathematical description which shows satisfying agreement with measurements, is at the same time simple enough to allow at least some analytical investigation of the problem, and is universal enough to apply not just to a particular material but also to a wider range of materials. In general theory of elasticity it is clear that simplicity of the Hooke's model (linear) allows development of rigorous and sophisticated theory which is on the other hand limited to certain basic assumptions. In contrast, the Ludwick's model (nonlinear) which is a generalization of the Hooke's model embraces a description of elastic behavior of a wider range of materials but is not so mathematically compliant. Beside that, it has a major deficiency. Namely, the stress gradient is infinite (or zero) for sufficiently small strains. This can be surpassed in a three-parametric generalized Ludwick's rheological model which is presented and used in this study of large deflections of non-prismatic cantilever beams.

From a practical standpoint, results obtained in this paper illustrate several advantages of the generalized Ludwick's model. We have developed an exact moment-curvature formula for materials which obey the generalized Ludwick's law. In addition, this formula can be applied to study arbitrarily loaded and supported beams. Moreover, numerical examples of our materially and geometrically nonlinear analysis clearly indicate rich nonlinear behavior of the discussed cantilever beams. It was also noticed that up to some characteristic strain the Ludwick type material behaves as harder than Hooke's material. After this strain, which was determined from the elastic strain energies of both materials, the Ludwick type material behaves as the softer one.

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