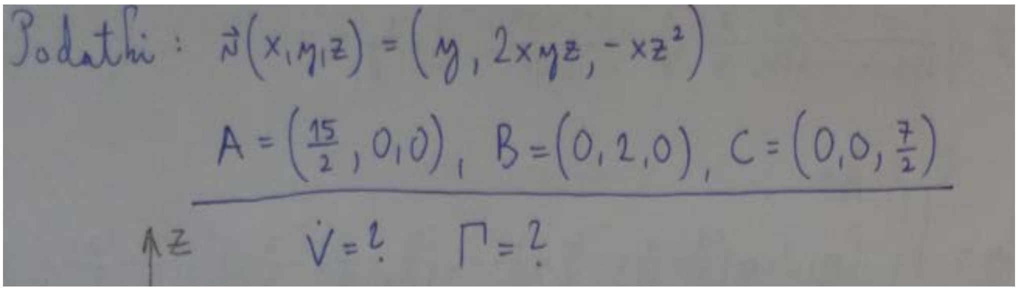
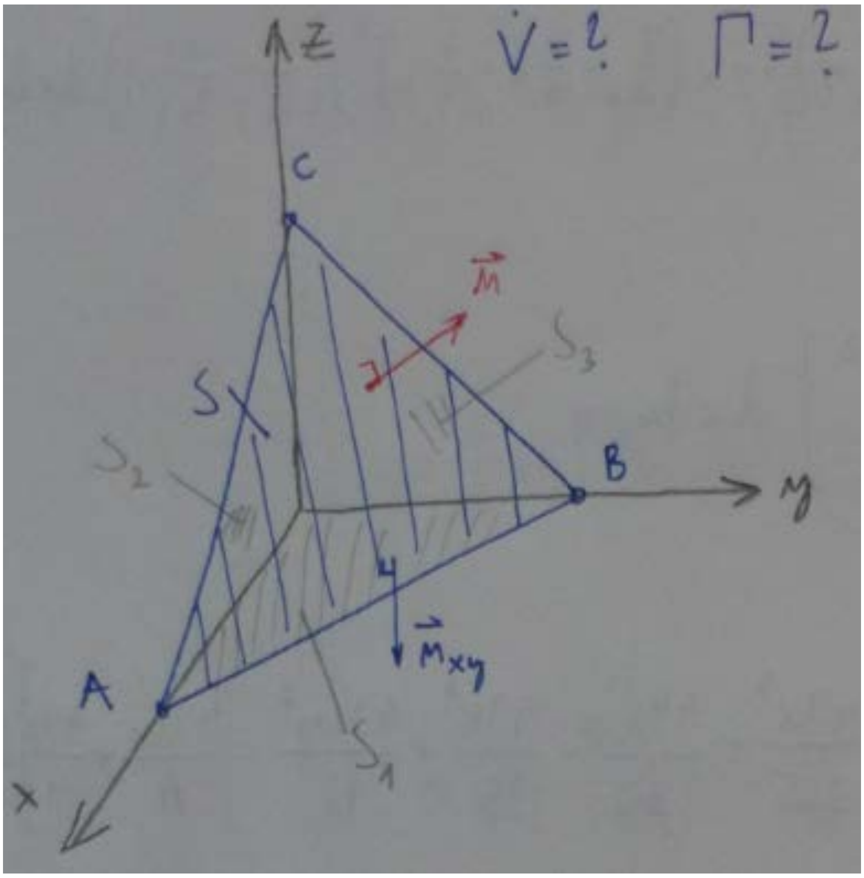
Naloga 1. Hitrostno polje je podano v m/s, položaji oglišč pa v m. Določi pretok in cirkulacijo na ploskvi, ki jo razpenjajo točke A, B in C.


Podatki: $\vec{v}(x,y,z) = (y, 2xyz, -xz^2)$

$A = \left(\frac{15}{2}, 0, 0\right)$, $B = (0, 2, 0)$, $C = \left(0, 0, \frac{7}{2}\right)$

$\dot{V} = ?$ $\Gamma = ?$



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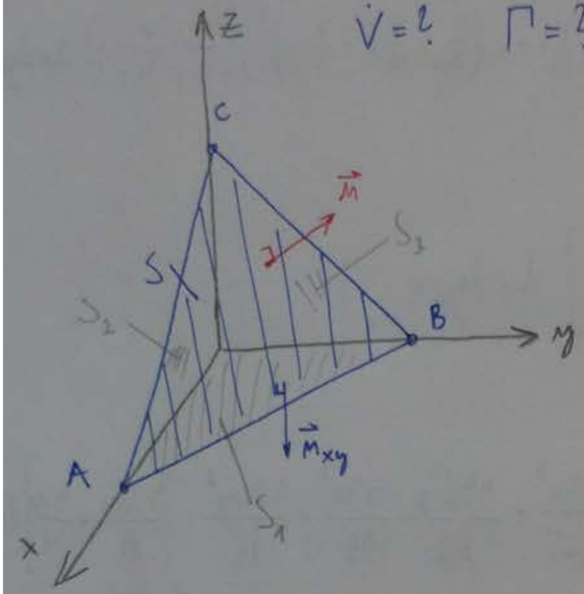


1. Način - integral prek S:

$$\dot{V} = \iint_S \vec{v} \cdot \vec{n}\, dA$$

→ Enačba ravnine:

$$z(x,y) = f(x,y) = \frac{7}{2} - \frac{7}{15}x - \frac{7}{4}y$$

→ Enotska normala na ravnino:

$$\vec{n} = \frac{\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right)}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1^2}} = \frac{\left(\frac{7}{15}, \frac{7}{4}, 1\right)}{\sqrt{\left(\frac{7}{15}\right)^2 + \left(\frac{7}{4}\right)^2 + 1^2}} = \frac{\left(\frac{7}{15}, \frac{7}{4}, 1\right)}{\sqrt{\frac{49}{225} + \frac{49}{16} + 1}} = \cancel{\sqrt{19858}}$$

$$= \sqrt{\frac{3600}{15409}}\left(\frac{7}{15}, \frac{7}{4}, 1\right)$$

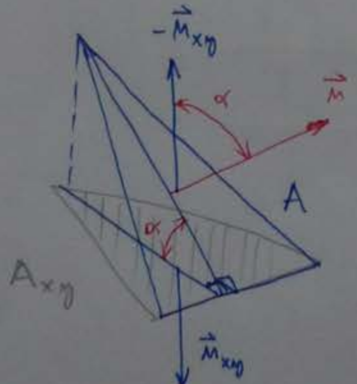
$dA_{x-y} = dx\,dy$ ... diferencial v ravnini x-y

$\vec{n}_{xy}$ ... normala na ravnino x-y

→ Določitev diferenciala dA:

Uporabimo formulo: $dA = \frac{dA_{xy}}{-\vec{n}_{xy} \cdot \vec{n}} = \frac{dx\,dy}{-(0,0,-1)\cdot\left(\frac{7}{15}, \frac{7}{4}, 1\right)\sqrt{\frac{3600}{15409}}} = \sqrt{\frac{15409}{3600}}\, dx\,dy$

↳ Izpeljava uporabljene formule:



$A_{xy} = A \cos\alpha$ ... projekcija ploskve

$$\cos\alpha = \frac{-\vec{n}_{xy} \cdot \vec{n}}{\underbrace{|-\vec{n}_{xy}|\,|\vec{n}|}_{=1\cdot 1}} = -\vec{n}_{xy} \cdot \vec{n}$$

$$A = \frac{A_{xy}}{\cos\alpha} = \frac{A_{xy}}{-\vec{n}_{xy} \cdot \vec{n}}$$

$$\dot V = \iint\limits_S \vec N\cdot\vec n\, dA = \iint \vec N \cdot \sqrt{\frac{36\cancel{00}}{\cancel{15409}}}\left(\frac{7}{15},\frac{7}{4},1\right)\cdot\sqrt{\frac{\cancel{15409}}{\cancel{3600}}}\, dx\, dy =$$

$$= \iint\limits_S \vec N\cdot\left(\frac{7}{15},\frac{7}{4},1\right) dx\, dy = \int_0^{\frac{15}{2}}\int_0^{2-\frac{4}{15}x}\left(y, 2xyz(x,y), -xz^2(x,y)\right)\cdot\left(\frac{7}{15},\frac{7}{4},1\right) dx\, dy =$$

$$= \int_0^{\frac{15}{2}}\int_0^{2-\frac{4}{15}x}\left[\frac{7}{15}y + \frac{7}{2}xy\left(\frac{7}{2}-\frac{7}{15}x-\frac{7}{4}y\right) - x\left(\frac{7}{2}-\frac{7}{15}x-\frac{7}{4}y\right)^2\right] dx\, dy$$

$$= \int_0^{\frac{15}{2}}\int_0^{2-\frac{4}{15}x}\left[-\frac{147}{16}xy^2 + \left(-\frac{49}{15}x^2+\frac{49}{2}x+\frac{7}{15}\right)y - \frac{49}{4}x - \frac{49}{225}x^3\right] dy\, dx =$$

$$= \int_0^{\frac{15}{2}}\left[-\frac{147}{16}x\frac{y^3}{3} + \left(-\frac{49}{15}x^2+\frac{49}{2}x+\frac{7}{15}\right)\frac{y^2}{2} - \left(\frac{49}{4}x+\frac{49}{225}x^3\right)y\right]_0^{2-\frac{4}{15}x} dx =$$

$$= \int_0^{\frac{15}{2}} \frac{56}{3375}x^2 - \frac{56}{225}x + \frac{14}{15}\, dx = \left[\frac{56}{3375}\frac{x^3}{3} - \frac{56}{225}\frac{x^2}{2} + \frac{14}{15}x\right]_0^{\frac{15}{2}} = \boxed{2{,}333\ \frac{m^3}{s}}$$

2. Način – divergenčni izrek:

$$\underline{\dot V_\Sigma = \dot V_S + \dot V_{S_1} + \dot V_{S_2} + \dot V_{S_3} = 0}$$

$$\dot V_\Sigma = \iint\limits_{S_\Sigma}\vec N\cdot\vec n\, dA \overset{\text{Gaussov izrek}}{=} \iiint\limits_V \vec\nabla\cdot\vec N\, dV = \iiint\limits_V \frac{\partial}{\partial x}N_x + \frac{\partial}{\partial y}N_y + \frac{\partial}{\partial z}N_z\, dV =$$

$$= \iiint\limits_V \frac{\partial}{\partial x}y + \frac{\partial}{\partial y}2xyz + \frac{\partial}{\partial z}\left(-xz^2\right) dV = \iiint\limits_V (2xz - 2xz)\, dV = \underline{0}$$

$$\dot{V}_{S_1} = \iint\limits_{S_1} \vec{N}\cdot\vec{m}_1\, dA_1 = \iint\limits_{S_1} (y, 2xyz, -xz^2)\cdot(0,0,-1)\, dx\, dy = \iint\limits_{S_1} x\overset{z=0}{z^2}(x,y)\, dx\, dy = 0$$

$$\dot{V}_{S_2} = \iint\limits_{S_2} \vec{N}\cdot\vec{m}_2\, dA_2 = \iint\limits_{S_2} (y, 2xyz, -xz^2)\cdot(0,-1,0)\, dx\, dz = \iint\limits_{S_2} \cancel{-2xyz} - 2xz\overset{y=0}{y}(x,z)\, dx\, dz = 0$$

$$\dot{V}_{S_3} = \iint\limits_{S_3} \vec{N}\cdot\vec{m}_3\, dA_3 = \iint\limits_{S_3} (y, 2xyz, -xz^2)(-1,0,0)\, dy\, dz = -\iint\limits_{S_3} y(z)\, dy\, dz =$$

$$= \cancel{\int_0^{\frac{7}{2}} dz \int_0^{\frac{7}{2}-\frac{4}{7}y}} = -\int_0^{\frac{7}{2}} dz \int_0^{2-\frac{4}{7}z} y\, dy = -\int_0^{\frac{7}{2}} \left[\frac{y^2}{2}\right]_0^{2-\frac{4}{7}z} dz = \cancel{-\frac{7}{2}\int_0^{\frac{7}{2}}\left(2-\frac{4}{7}z\right)} =$$

$$= -\frac{1}{2}\int_0^{\frac{7}{2}} \left(2-\frac{4}{7}z\right)^2 dz = -\frac{1}{2}\int_0^{\frac{7}{2}} \left(\frac{16}{49}z^2 - \frac{16}{7}z + 4\right) dz = -\frac{1}{2}\left[\frac{16}{49}\frac{z^3}{3} - \frac{16}{7}\cdot\frac{z^2}{2} + 4z\right]_0^{\frac{7}{2}} =$$

$$= \underline{-2{,}333\ \frac{m^3}{s}} \rightarrow \dot{V}_S = \dot{V}_\Sigma - \dot{V}_{S_1} - \dot{V}_{S_2} - \dot{V}_{S_3} = 0 - 0 - 0 - \left(-2{,}333\ \frac{m^3}{s}\right) = \boxed{2{,}333\ \frac{m^3}{s}}$$

Stokesov izrek

$$\Gamma = \oint\limits_{\partial S} \vec{N}\cdot\vec{t}\, ds = \iint\limits_{S} (\vec{\nabla}\times\vec{N})\cdot\vec{m}\, dA = \iint\limits_{S} \left(\frac{\partial N_z}{\partial y} - \frac{\partial N_y}{\partial z}, \frac{\partial N_x}{\partial z} - \frac{\partial N_z}{\partial x}, \frac{\partial N_y}{\partial x} - \frac{\partial N_x}{\partial y}\right)\cdot\vec{m}\, dA =$$

$$= \int_0^{\frac{15}{2}}\int_0^{2-\frac{4}{15}x} \left(2xy, -z^2(x,y), 2yz(x,y) - 1\right)\cdot\left(\frac{7}{15}, \frac{7}{4}, 1\right) dx\, dy =$$

$$= \int_0^{\frac{15}{2}}\int_0^{2-\frac{4}{15}x} \left[\frac{14}{15}xy - \frac{7}{4}\left(\frac{7}{2} - \frac{7}{15}x - \frac{7}{4}y\right)^2 + 2y\left(\frac{7}{2} - \frac{7}{15}x - \frac{7}{4}y\right) - 1\right] dx\, dy =$$

$$= \int_0^{\frac{15}{2}} \left[\frac{7}{125}x^3 - \frac{91}{90}x^2 + \frac{359}{60}x - \frac{93}{8}\right] dx = -16{,}7969\ \frac{m^2}{s} \doteq \boxed{-16{,}80\ \frac{m^2}{s}}$$

Naloga 2. Točke A, B in C definirajo ravnino v prostoru. Hitrostno polje je podano v m/s, položaji točk pa v m. Določi vrednosti konstant a, b in c, da bo hitrostno polje opisovalo tok nestisljivega fluida. Nadalje določi pretok in cirkulacijo na ploskvi, ki jo omejujejo daljice med A, B in C.

Podatki: $\vec{v}(x,y,z) = \left(c y^2 + 5z,\ 6y(x-z),\ a y^2 + b x\right)$

$A = (5,0,0)$, $B = (0,3,0)$, $C = (0,0,8)$

$\dot{V} = ?$ $\quad \Gamma = ?$